The Annals of Probability
2001, Vol. 29, No. 1, 123–126

A CORRELATION INEQUALITY FOR CONNECTION EVENTS IN PERCOLATION

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It is well-known in percolation theory (and intuitively plausible) that two events of the form "there is an open path from s to a" are positively correlated. We prove the (not intuitively obvious) fact that this is still true if we condition on an event of the form "there is no open path from s to t."

1. Introduction and statement of results. We consider the usual bond percolation models on a (finite or countably infinite) graph $G = (V, E)$: each $e \in E$ is "open" (has value 1) with probability $p(e)$ and "closed" (has value 0) with probability $1 - p(e)$, independently of all other edges. We write P for the corresponding probability distribution on $\Omega := \{0, 1\}^E$. For general background see [4].

For $s, a \in V$ we write $s \leftrightarrow a$ for the event that there is an open path from s to a, and $s \not\leftrightarrow a$ for the complementary event.

Positive (i.e., nonnegative) correlation of any two events $s \leftrightarrow a$ and $s \leftrightarrow b$ follows from Harris' inequality [5] (Theorem 2.1 below). The correlation inequality of the title says that this phenomenon persists if we condition on any event $s \not\leftrightarrow t$.

THEOREM 1.1. *For any* $s, a, b, t \in V$,

$$P(s \leftrightarrow a, s \leftrightarrow b | s \not\leftrightarrow t) \geq P(s \leftrightarrow a | s \not\leftrightarrow t) P(s \leftrightarrow b | s \not\leftrightarrow t).$$

The intuition for this is not very clear. In particular it is *not* true if we condition on $s \leftrightarrow t$ rather than $s \not\leftrightarrow t$. (Consider the graph with vertices s, a, b, t and each of s, t joined to each of a, b.)

From now on we fix $s \in V$, and set, for $X \subseteq V$, $Q_X = \{s \leftrightarrow x \, \forall \, x \in X\}$ and $R_X = \{s \not\leftrightarrow x \, \forall \, x \in X\}$.

THEOREM 1.2. *For any* $A, B, X, Y \subseteq V$,

$$P(Q_A R_X) P(Q_B R_Y) \leq P(Q_{A \cup B} R_{X \cap Y}) P(R_{X \cup Y}). \tag{1}$$

REMARKS 1. Of course we recover Theorem 1.1 from Theorem 1.2 by taking $A = \{a\}$, $B = \{b\}$ and $X = Y = \{t\}$. This is not generalization for its own sake: the more general form is needed for the proof.

Received July 1999; revised June 2000.
[1]Supported by NSF Grant DMS-9970433.
AMS 2000 *subject classifications.* 05C99, 60C05, 60K35
Key words and phrases. Correlation inequalities, percolation, Ahlswede–Daykin theorem.

2. The perhaps intuitively more natural statement obtained by replacing $R_{X\cup Y}$ by $Q_{A\cap B}R_{X\cup Y}$ in Theorem 1.2 is *not* true: take $V(G) = \{s, x, y, a\}$, $E(G) = \{sx, xa, ay, ys\}$ and $X = \{x\}$, $Y = \{y\}$, $A = B = \{a\}$.
3. As pointed out to us by the referee, Theorem 1.1 can be generalized to sets of vertices S, A, B, T by replacing s by $S, \dots,$ t by T, and interpreting $X \leftrightarrow Y$ as $\{\exists x \in X, y \in Y x \leftrightarrow y\}$. To see this, simply identify all vertices in each of S, A, B, T, retaining multiple edges, and apply Theorem 1.1.
4. Note that if we replace A by $A\backslash B$ in Theorem 1.2, the r.h.s. of (1) remains the same and the l.h.s. does not decrease. So Theorem 1.2 as stated above is not more general than the case $A \cap B = \emptyset$.
5. The original motivation for Theorem 1.1 was a conjecture we learned from the late P. W. Kasteleyn (personal communication, circa 1985), a slightly informal description of which is as follows. Let $G = (V, E)$ be a finite graph, W some subset of V and $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ a copy of G. For each $e \in E$ and $v \in V$, let $\tilde{e}$ and $\tilde{v}$ be the corresponding edge and vertex in $\widetilde{G}$, respectively. Now we "glue" G and $\widetilde{G}$ together by identifying w with $\tilde{w}$ for $w \in W$, and on this new graph consider any percolation model with $p(\tilde{e}) = p(e)$ for all $e \in E$. The conjecture is then that, for every $a, b \in V$, $P(a \leftrightarrow b) \geq P(a \leftrightarrow \tilde{b})$. There is in fact a slight concrete connection with Theorem 1.1, in that a special case of the latter says that when $|W| = 2$, say $W = \{v, w\}$, one has $P(a \leftrightarrow b | v \not\leftrightarrow w) \geq P(a \leftrightarrow \tilde{b} | v \not\leftrightarrow w)$. But we feel that Theorem 1.1 is more interesting for its own sake and believe it has potential applications in percolation theory in general.

2. Background. We just recall the two correlation inequalities we will need in Section 3. For more extensive discussions see [2].

An event $\mathscr{A}$ (i.e., a subset of Ω) is called *increasing* if $\mathscr{A} \ni \omega \leq \omega'$ implies $\omega' \in A$. (Here $\omega \leq \omega'$ means $\omega_e \leq \omega'_e$ for all $e \in E$.) The following correlation inequality is due to Harris [5].

THEOREM 2.1. *For any increasing $\mathscr{A}, \mathscr{B} \subset \Omega$,*

$$P(\mathscr{A}\mathscr{B}) \geq P(\mathscr{A})P(\mathscr{B}).$$

Of course this is equivalent to saying that for any increasing $\mathscr{A}$ and *decreasing* $\mathscr{B}$, $P(\mathscr{A}\mathscr{B}) \leq P(\mathscr{A})P(\mathscr{B})$.

There are a number of significant extensions of Harris' inequality, notably that of Fortuin, Kasteleyn and Ginibre [3]. (We are informed by the referee that the inequality was essentially given a bit earlier in [6].) Our main tool is the considerably more general Ahlswede–Daykin (or "four functions") theorem [1], namely:

THEOREM 2.2. *Let N be a finite set and let $\mathscr{P}(N)$ denote the set of all subsets of N. Suppose $\alpha, \beta, \gamma, \delta$: $\mathscr{P}(N) \to \mathbf{R}^+$ satisfy*

$$\alpha(S)\beta(T) \leq \gamma(S \cap T)\delta(S \cup T) \qquad \forall\, S, T \subseteq N. \tag{2}$$

Then $\sum \alpha(S) \sum \beta(S) \leq \sum \gamma(S) \sum \delta(S)$ *(where the sums are over all* $S \subseteq N$*).*

3. Proof of Theorem 1.2. We assume G is finite. (If G is countably infinite, the result follows from the finite case by obvious limit arguments.) The proof is by induction on the number of vertices $|V|$. If $|V| = 1$, the result is trivial. Suppose it always holds if $|V| \leq n$ and consider a graph G with $n + 1$ vertices.

Set $X \cap Y = Z$. If $Z = \varnothing$ then (1) follows from the Harris inequality:

$$\begin{aligned} P(Q_A R_X)P(Q_B R_Y) &\leq P(Q_A)P(R_X)P(Q_B)P(R_Y) \\ &\leq P(Q_A Q_B)P(R_X R_Y) \\ &= P(Q_{A\cup B} R_{X\cap Y})P(R_{X\cup Y}). \end{aligned}$$

If $Z \neq \varnothing$ we proceed as follows: set $N = \{y \notin Z\colon y \sim Z\}$ (where $y \sim Z$ means y is adjacent to at least one vertex of Z). Define the (random) set

$$\mathbf{S} = \{y \in N\colon \text{there is an open edge from } y \text{ to } Z\}.$$

We use S, T for possible values of $\mathbf{S}$ and write $P(S)$ for $P(\mathbf{S} = S)$ and $P(\cdot|S)$ for the conditional distribution given $\mathbf{S} = S$. We may expand

$$P(Q_A R_X) = \sum_S P(S)P(Q_A R_X|S)$$

(where the sum is over all subsets of N) and similarly for the other terms in (1). Thus if we define

$$\begin{aligned} \alpha(S) &= P(S)P(Q_A R_X|S), \\ \beta(S) &= P(S)P(Q_B R_Y|S), \\ \gamma(S) &= P(S)P(Q_{A\cup B} R_{X\cap Y}|S), \\ \delta(S) &= P(S)P(R_{X\cup Y}|S), \end{aligned}$$

then (1) becomes

$$\sum \alpha(S) \sum \beta(S) \leq \sum \gamma(S) \sum \delta(S),$$

where S runs over the subsets of N. Theorem 2.2 says that to verify this we just need to establish (2), which, since (as one can easily check) $P(S)P(T) = P(S \cup T)P(S \cap T)$, is the same as

$$P(Q_A R_X|S)P(Q_B R_Y|T) \leq P(Q_{A\cup B} R_{X\cap Y}|S \cap T)P(R_{X\cup Y}|S \cup T). \tag{3}$$

Let P' refer to the percolation model for the graph G', obtained from G by removing Z, with edge probabilities as in our original percolation model on G. Then it is easy to see that for any $C, W \subseteq V\backslash Z$ and $S \subseteq N$,

$$P(Q_C R_{W\cup Z}|S) = P'(Q_C R_{W\cup S}). \tag{4}$$

Now we obtain (3) as follows: Let $X' = X\backslash Z$ and $Y' = Y\backslash Z$. We have

$$\begin{aligned} P(Q_A R_X|S)P(Q_B R_Y|T) &= P'(Q_A R_{X'\cup S})P'(Q_B R_{Y'\cup T}) \\ &\le P'(Q_{A\cup B} R_{(X'\cup S)\cap(Y'\cup T)})P'(R_{(X'\cup S)\cup(Y'\cup T)}) \\ &\le P'(Q_{A\cup B} R_{(S\cap T)})P'(R_{(X'\cup Y')\cup(S\cup T)}) \\ &= P(Q_{A\cup B} R_{X\cap Y}|S\cap T)P(R_{X\cup Y}|S\cup T), \end{aligned}$$

where the first equality follows from applying (4) twice (with $W = X'$ and $W = Y'$, respectively), the first inequality from the induction hypothesis [which says that (1) holds for G'], the second inequality from $(S \cap T) \subseteq (X' \cup S) \cap (Y' \cup T)$, and the second equality from again applying (4) twice (with $W = \emptyset$ and $W = X' \cup Y'$, respectively). □

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